

# An intermediate level of approximation for computing the dual descriptor

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**Abstract** At present, there are two levels of approximation to compute the dual descriptor (DD). The first uses the total electronic density of the original system along with the electronic densities of the system with one more electron and one less electron, but this procedure is time consuming and normal termination of computation of total electronic densities is not guaranteed. The second level of approximation uses only the electronic densities of frontier molecular orbitals, HOMO and LUMO, to avoid the former approximation; however, the orbital relaxation implicitly included in the first level of approximation is absent in the second, thus risking an incorrect interpretation of local reactivity. Between the lowest occupied molecular orbital (LOMO) and the highest unoccupied molecular orbital (HUMO), a framework to provide an expression of the DD in terms of the electronic densities of all molecular orbitals (except HUMO and LOMO) has been proposed to be implemented by programmers as a computational code. This methodology implies another level of approximation located between the conventional approximation methods mentioned above. In this study, working equations have been oriented toward molecular closed- and open-shell systems. In addition, the mathematical expression for a closed-shell system was applied to acetylene in order to assess the capability of this approach to generate the DD.

**Keywords** Local reactivity · Dual descriptor · Nucleophilic Fukui function · Electrophilic Fukui Function · Expansion in terms of electronic densities of molecular orbitals · Closed-shell system · Open-shell system

## Introduction

When chemical reactions are governed by covalent rather than by electrostatic interactions, a local reactivity descriptor (LRD) coming from conceptual density functional theory called the Fukui function [1–4] is recommended as a useful tool to find preferred sites on a molecule to donate or capture electrons. The Fukui function is defined in terms of the derivative of  $\rho(\mathbf{r})$  with respect to  $N$ ; through a Maxwell relation, the same descriptor is interpreted as the variation of the chemical potential,  $\mu$ , with respect to the variation of the external potential,  $v(\mathbf{r})$ :

$$f(\mathbf{r}) = \left( \frac{\partial \rho(\mathbf{r})}{\partial N} \right)_{v(\mathbf{r})} = \left[ \frac{\delta \mu}{\delta v(\mathbf{r})} \right]_N. \quad (1)$$

Several higher-order reactivity descriptors have been established [5]. The chemical potential  $\mu$  characterizes the tendency of electrons to escape from the equilibrium system [6–8]; meanwhile, the external potential,  $v(\mathbf{r})$  [8, 9] is created by a specific number of nuclei with their respective electric charges. The function  $f(\mathbf{r})$  reflects the ability of a molecular site to accept or donate electrons. High values of  $f(\mathbf{r})$  are related to high reactivity at point  $\mathbf{r}$  [8, 9]. In addition, the Fukui function must satisfy the following normalization condition:

$$\int_{\text{all space}} f(\mathbf{r}) d\mathbf{r} = 1. \quad (2)$$

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From Eq. (1), if paying attention to  $\rho(\mathbf{r})$  and its dependence on  $N$  rather than to  $\mu$  and its dependence on  $v(\mathbf{r})$ , the analysis will center on electronic densities and not on chemical potentials. Since the number of electrons  $N$  is a discrete variable [10], a first level of approximation called finite difference approximation (FDA) implies the use of right and left derivatives of  $\rho(\mathbf{r})$  with respect to  $N$  to compute Eq. (1), thus providing a couple of working equations:

$$f^+(\mathbf{r}) = \left( \frac{\partial \rho(\mathbf{r})}{\partial N} \right)_{v(\mathbf{r})}^+ = \rho_{N+1}(\mathbf{r}) - \rho_N(\mathbf{r}) \quad (3)$$

$$f^-(\mathbf{r}) = \left( \frac{\partial \rho(\mathbf{r})}{\partial N} \right)_{v(\mathbf{r})}^- = \rho_N(\mathbf{r}) - \rho_{N-1}(\mathbf{r}) \quad (4)$$

where  $\rho_{N+1}(\mathbf{r})$ ,  $\rho_N(\mathbf{r})$  and  $\rho_{N-1}(\mathbf{r})$  are the electronic densities at point  $\mathbf{r}$  for a system with  $N+1$ ,  $N$  and  $N-1$  electrons, respectively. Although the finite difference approximation is exact for the exact density functional as Levy pointed out [11], this mathematical procedure is hereafter referred to as a first level of approximation because is not strictly speaking a derivative due to the discrete nature of  $N$ , thus requiring the use of a numerical derivative from the left and right. The first numerical derivative,  $f^+(\mathbf{r})$ , is associated with reactivity for a nucleophilic attack so that it measures the intramolecular reactivity at site  $\mathbf{r}$  toward a nucleophilic reagent. The second,  $f^-(\mathbf{r})$ , is associated with reactivity for an electrophilic attack so that this function measures the intramolecular reactivity at site  $\mathbf{r}$  toward an electrophilic reagent [12]. Nevertheless, this set of working equations does not properly describe the local reactivity in highly symmetric molecules. Hence, this article attempts to establish a correct description of local reactivity based on this level of approximation. For open-shell systems, a working equation based on the first level of approximation has been adapted to the molecules' symmetry [13].

The second level of approximation implies the use of densities of frontier molecular orbitals (FMOs),  $\rho_{\text{LUMO}}(\mathbf{r})$  (LUMO density) and  $\rho_{\text{HOMO}}(\mathbf{r})$  (HOMO density), which are taken into account because it has been shown [12, 14] that when orbital relaxation is irrelevant, there is a direct relation between  $f^{+/-}(\mathbf{r})$  and the density of the appropriate FMO, thus avoiding calculations of the system with  $N+1$  and  $N-1$  electrons. This is the frontier molecular orbital approximation (FMOA) also called the frozen orbital approximation (FOA):

$$f^+(\mathbf{r}) \simeq \rho_{\text{LUMO}}(\mathbf{r}), \quad (5)$$

$$f^-(\mathbf{r}) \simeq \rho_{\text{HOMO}}(\mathbf{r}) \quad (6)$$

The use of the two working Eqs. (5) and (6) which are based on electronic densities of FMOs, should always be checked against the working Eqs. (3) and (4) coming from the first level of approximation.

More recently, Morell et al. [15, 16] found a third order, i.e., a physical interpretation for a new LRD [17] for chemical reactivity called dual descriptor (DD)  $\Delta f(\mathbf{r})$ . To avoid any confusion with finite difference approximation techniques, the old notation,  $\Delta f(\mathbf{r})$ , will be dropped and the modern notation,  $f^{(2)}(\mathbf{r})$ , will be used hereafter.

This LRD is defined in terms of the derivative of  $f(\mathbf{r})$  with respect to  $N$ . Through a Maxwell relation, the same descriptor is interpreted as the variation of  $\eta$  with respect to  $v(\mathbf{r})$ . The definition of  $f^{(2)}(\mathbf{r})$  is shown as indicated by Morell et al. [15, 16]:

$$f^{(2)}(\mathbf{r}) = \left( \frac{\partial f(\mathbf{r})}{\partial N} \right)_{v(\mathbf{r})} = \left[ \frac{\delta \eta}{\delta v(\mathbf{r})} \right]_N, \quad (7)$$

where  $\eta$  is the molecular hardness, which measures the resistance to charge transfer [2]. The DD satisfies the following restriction:

$$\int_{\text{all space}} f^{(2)}(\mathbf{r}) d\mathbf{r} = 0. \quad (8)$$

Additionally, and according to expressions given by Eqs. (3) and (4),  $f^{(2)}(\mathbf{r})$  is written as the difference between nucleophilic and electrophilic Fukui functions [15]:

$$f^{(2)}(\mathbf{r}) \simeq f^+(\mathbf{r}) - f^-(\mathbf{r}) = \rho_{N+1}(\mathbf{r}) - 2\rho_N(\mathbf{r}) + \rho_{N-1}(\mathbf{r}). \quad (9)$$

The use of densities of FMOs provides an easier-to-compute working equation:

$$f^{(2)}(\mathbf{r}) \simeq \rho_{\text{LUMO}}(\mathbf{r}) - \rho_{\text{HOMO}}(\mathbf{r}). \quad (10)$$

The DD, by means of Eq. (9) or Eq. (10), allows one to obtain simultaneously the preferred sites for nucleophilic attacks ( $f^{(2)}(\mathbf{r}) > 0$ ) and the preferred sites for electrophilic attacks ( $f^{(2)}(\mathbf{r}) < 0$ ) over the system at point  $\mathbf{r}$ .

This LRD has acquired more importance because it has demonstrated to be a robust tool with which to correctly predict specific sites of nucleophilic and electrophilic attacks much more efficiently than the Fukui function by itself because the DD distinguishes true nucleophilic and electrophilic sites. Several published works have noted the outstanding capability of  $f^{(2)}(\mathbf{r})$  and all LRDs based on it [18–22]. Molecular symmetry affects local reactivity and, as a consequence, the Fukui function should conserve this symmetry, as shown by Flores-Moreno [23]. Previously, Martínez [24] proposed a simple procedure to take into account the frontier molecular orbital degeneracy when using the FOA [12, 14], so that  $f^{(2)}(\mathbf{r})$  and any other local reactivity descriptor depending on the DD may be properly depicted as a 3-D map, thus allowing a local reactivity isosurface adapted to the molecular symmetry to be

obtained. A recent demonstration of this was published in an article [25] in which the local reactivity of the Buckminster fullerene was established by taking into account the degeneracy in FMO mentioned above. A concern about evaluation of the DD was broached by Geerlings and coworkers [26–28], and the problem of degeneracy in the Fukui function and DD was analyzed by Cárdenas and coworkers [29] in the context of degenerate ground states.

However, both levels of approximation, although adapted to a molecules' symmetry, apply extreme models to face the problem of depicting the DD. On the one hand, the use of Eqs. (3) and (4) provides the most accurate result possible, but at very high computational cost and without knowing whether convergence will be reached or not for a system with one more electron and one less electron; on the other hand, the use of Eqs. (5) and (6) leads to a faster result, but with the intrinsic risk of misunderstanding the information generated due to the simple approach of assuming that local reactivity is guided only by the density of FMOs and thus neglecting the possible effects of orbital relaxation. The problem of taking orbital relaxation into account has been broached by Ayers [30–32].

The present study proposes a couple of working equations for closed-shell and open-shell systems, which may be understood as an extension of the second level of approximation in order to compute DD based on densities of FMOs plus the contribution of densities of the remaining molecular orbitals, both occupied and virtual. Taking this approach increased the computational effort a little, but without including the risk of performing quantum mechanical computations for a system with one more electron and one less electron. Also, it includes the influence of the remaining molecular orbitals over local reactivity. The latter may be interpreted as an approximation to quantify orbital relaxation effects.

### Dual descriptor expanded in terms of electronic densities of molecular orbitals. Closed-shell systems

The following is based on articles published by Politzer and coworkers [33–35], where the sum of the electronic density weighted molecular orbital energies produced a concept called average local ionization energy (ALIE). Assuming a basis set of  $K$  molecular orbitals for a closed-shell system in the ground state, and assuming that there is no degeneration among them, then  $N/2$  occupied molecular orbitals emerge, following the order of lowest energy to highest energy (Fig. 1). The  $i$  counter will range from 1 (LOMO) to  $N/2$  (HOMO). Immediately, it can be deduced that  $N/2+1$  will be the LUMO, and, after  $i=N/2$ , the counter  $i$  will continue

ranging from  $N/2+1$  (LUMO) to  $K$  (HUMO). On the other hand, each molecular orbital  $\psi_i(\mathbf{r})$  implies an electronic density  $\rho_{[i]}(\mathbf{r}) \equiv |\psi_i(\mathbf{r})|^2$ , therefore  $\rho_{[i]}(\mathbf{r})$  stands for the density of the  $i^{th}$ -molecular orbital.

Note that the squared parenthesis in the subscript of  $\rho_{[i]}(\mathbf{r})$  has been used in order to avoid any confusion between  $\rho_N(\mathbf{r})$ , the total electronic density of the system conformed by  $N$  electrons and  $\rho_{[N]}(\mathbf{r})$ , the electronic density of the  $N^{th}$ -molecular orbital. Note that  $\rho_N(\mathbf{r}) = 2 \sum_{i=1}^{N/2} |\psi_i(\mathbf{r})|^2 \equiv 2 \sum_{i=1}^{N/2} \rho_{[i]}(\mathbf{r})$  where the coefficient 2 corresponds to the occupation number; it follows that:

$$\int \rho_{[i]}(\mathbf{r}) d\mathbf{r} = 1; \forall i \in \{1 \dots K\}$$

### Nucleophilic Fukui function as an expansion of electronic densities of virtual molecular orbitals

The following expression is proposed as a Fukui nucleophilic function. The energy of HUMO,  $\varepsilon_{[K]}$ , is used as a reference value:

$$f^+(\mathbf{r}) \simeq \sum_{i=N/2+1}^{K-1} \chi_{[i]}^+ \rho_{[i]}(\mathbf{r}) \tag{11}$$

where

$$\chi_{[i]}^+ = (\varepsilon_{[K]} - \varepsilon_{[i]}) \cdot \left\{ \sum_{j=N/2+1}^{K-1} (\varepsilon_{[K]} - \varepsilon_{[j]}) \right\}^{-1}$$

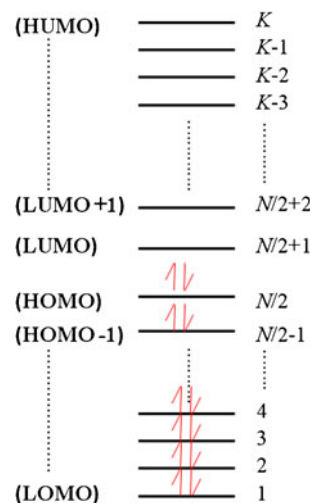


Fig. 1 Diagram of molecular orbital energies in a closed-shell system

$\varepsilon_{[i]}$  and  $\varepsilon_{[j]}$  stands for the energy of the  $i^{\text{th}}$ -molecular orbital and  $j^{\text{th}}$ -molecular orbital, respectively. The reader should note that each coefficient  $\chi_{[i]}^+$  satisfies the condition  $0 < \chi_{[i]}^+ < 1$  which leads to the natural restriction for the sum of all coefficients with  $i$  ranging from  $N/2 + 1$  to  $K-1$ :

$$\sum_{i=N/2+1}^{K-1} \chi_{[i]}^+ = 1.$$

This satisfies the normalization condition of the Fukui function given by Eq. (2).

Eq. (11) is written explicitly as follows:

$$f^+(\mathbf{r}) \simeq \sum_{i=N/2+1}^{K-1} \chi_{[i]}^+ \rho_{[i]}(\mathbf{r})$$

$$f^+(\mathbf{r}) \simeq \frac{\varepsilon_{[K]} - \varepsilon_{[N/2+1]}}{\left(\varepsilon_{[K]} - \varepsilon_{[N/2+1]}\right) + \left(\varepsilon_{[K]} - \varepsilon_{[N/2+2]}\right) + \dots + \left(\varepsilon_{[K]} - \varepsilon_{[K-1]}\right)} \cdot \rho_{[N/2+1]}(\mathbf{r}) + \dots$$

$$\dots + \frac{\varepsilon_{[K]} - \varepsilon_{[K-1]}}{\left(\varepsilon_{[K]} - \varepsilon_{[N/2+1]}\right) + \left(\varepsilon_{[K]} - \varepsilon_{[N/2+2]}\right) + \dots + \left(\varepsilon_{[K]} - \varepsilon_{[K-1]}\right)} \cdot \rho_{[K-1]}(\mathbf{r})$$

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$$f^+(\mathbf{r}) \simeq \sum_{i=N/2+1}^{K-1} \chi_{[i]}^+ \rho_{[i]}(\mathbf{r})$$

$$\simeq \sum_{i=N/2+1}^{N/2+p} \chi_{[i]}^+ \rho_{[i]}(\mathbf{r}) + \sum_{i=N/2+p+1}^{K-1} \chi_{[i]}^+ \rho_{[i]}(\mathbf{r})$$

$$\simeq \frac{\varepsilon_{[K]} - \varepsilon_{[N/2+1]}}{\sum_{j=N/2+1}^{K-1} \left(\varepsilon_{[K]} - \varepsilon_{[j]}\right)} \cdot \rho_{[N/2+1]}(\mathbf{r}) + \dots + \frac{\varepsilon_{[K]} - \varepsilon_{[N/2+p]}}{\sum_{j=N/2+1}^{K-1} \left(\varepsilon_{[K]} - \varepsilon_{[j]}\right)} \cdot \rho_{[N/2+p]}(\mathbf{r}) + \sum_{i=N/2+p+1}^{K-1} \chi_{[i]}^+ \rho_{[i]}(\mathbf{r})$$

$$\simeq \frac{\varepsilon^+}{\sum_{j=N/2+1}^{K-1} \left(\varepsilon_{[K]} - \varepsilon_{[j]}\right)} \cdot \rho_{[N/2+1]}(\mathbf{r}) + \dots + \frac{\varepsilon^+}{\sum_{j=N/2+1}^{K-1} \left(\varepsilon_{[K]} - \varepsilon_{[j]}\right)} \cdot \rho_{[N/2+p]}(\mathbf{r}) + \sum_{i=N/2+p+1}^{K-1} \chi_{[i]}^+ \rho_{[i]}(\mathbf{r})$$

$$\simeq \frac{\varepsilon^+}{\sum_{j=N/2+1}^{K-1} \left(\varepsilon_{[K]} - \varepsilon_{[j]}\right)} \cdot \sum_{i=N/2+1}^{N/2+p} \rho_{[i]}(\mathbf{r}) + \sum_{i=N/2+p+1}^{K-1} \chi_{[i]}^+ \rho_{[i]}(\mathbf{r})$$


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The denominator of the first term can be split into two parts:

$$\sum_{j=N/2+1}^{K-1} \left(\varepsilon_{[K]} - \varepsilon_{[j]}\right) = \sum_{j=N/2+1}^{N/2+p} \left(\varepsilon_{[K]} - \varepsilon_{[j]}\right) + \sum_{j=N/2+p+1}^{K-1} \left(\varepsilon_{[K]} - \varepsilon_{[j]}\right)$$

$$= p \cdot \varepsilon^+ + \sum_{j=N/2+p+1}^{K-1} \left(\varepsilon_{[K]} - \varepsilon_{[j]}\right)$$

The following relationship must always be satisfied:

$$1 > \frac{\varepsilon_{[K]} - \varepsilon_{[N/2+1]}}{\sum_{j=N/2+1}^{K-1} \left(\varepsilon_{[K]} - \varepsilon_{[j]}\right)} > \dots > \frac{\varepsilon_{[K]} - \varepsilon_{[K-1]}}{\sum_{j=N/2+1}^{K-1} \left(\varepsilon_{[K]} - \varepsilon_{[j]}\right)} > 0 \quad (12)$$

Unlike Eq. (5), and those adapted to the symmetry of molecules as in the respective literature [24], thanks to  $\chi_{[i]}^+$  coefficients, Eq. (11) is adapted intrinsically to the molecule's symmetry. In fact, the working equations just mentioned constitute approximations of Eq. (11) as will be demonstrated in the next paragraph. Thus, Eq. (5), together with that proposed in 2009 [24], correspond to approximations coming from Eq. (11). This is shown as follows:

Demonstration: imagine a molecular system with a degree of degeneration,  $p$ , in its virtual molecular orbitals; specifically, it presents a  $p$ -fold degenerated LUMO. That means:

$$\varepsilon_{[K]} - \varepsilon_{[N/2+1]} = \varepsilon_{[K]} - \varepsilon_{[N/2+2]} = \dots = \varepsilon_{[K]} - \varepsilon_{[N/2+p]} = \varepsilon^+.$$

As a consequence, the sum of Eq. (11) may be decomposed as follows:

According to the latter,

$$f^+(\mathbf{r}) \simeq \varepsilon^+ \cdot \left\{ \sum_{j=N/2+1}^{K-1} \left(\varepsilon_{[K]} - \varepsilon_{[j]}\right) \right\}^{-1} \cdot \sum_{i=N/2+1}^{N/2+p} \rho_{[i]}(\mathbf{r}) + \sum_{i=N/2+p+1}^{K-1} \chi_{[i]}^+ \rho_{[i]}(\mathbf{r})$$

$$\simeq \varepsilon^+ \cdot \left\{ p \cdot \varepsilon^+ + \sum_{j=N/2+p+1}^{K-1} \left(\varepsilon_{[K]} - \varepsilon_{[j]}\right) \right\}^{-1} \cdot \sum_{i=N/2+1}^{N/2+p} \rho_{[i]}(\mathbf{r}) + \sum_{i=N/2+p+1}^{K-1} \chi_{[i]}^+ \rho_{[i]}(\mathbf{r})$$

Based on inequality (12), in this last expression, consider that  $\chi_{[i]}^+ \simeq 0$  when  $i$  ranges from  $N/2 + p + 1$  to  $K-1$  along with  $p \cdot \varepsilon^+ \gg \sum_{j=N/2+p+1}^{K-1} (\varepsilon_{[K]} - \varepsilon_{[j]})$ . As a result, the equation used to obtain the nucleophilic Fukui function, adapted to the symmetry according to the degree of degeneration, is:

$$f^+(\mathbf{r}) \simeq \frac{1}{P} \sum_{i=N/2+1}^{N/2+p} \rho_{[i]}(\mathbf{r}) \equiv \frac{1}{P} \sum_{k=1}^p \rho_{\text{LUMO}_k}(\mathbf{r})$$

This equation, which was already suggested in 2009 [24], is an approximation of Eq. (11) in the case of a  $p$ -fold degenerated LUMO. When  $p=1$ , the classical Eq. (5) is recovered because  $\rho_{\text{LUMO}_1}(\mathbf{r}) \equiv \rho_{\text{LUMO}}(\mathbf{r})$ .

Electrophilic Fukui function as an expansion of electronic densities of occupied molecular orbitals

A similar analysis leads to the expression of the electrophilic Fukui function. The energy of LOMO,  $\varepsilon_{[1]}$ , is used as a reference value:

$$f^-(\mathbf{r}) \simeq \sum_{i=2}^{N/2} \chi_{[i]}^- \rho_{[i]}(\mathbf{r}) \tag{13}$$

where

$$\chi_{[i]}^- = (\varepsilon_{[i]} - \varepsilon_{[1]}) \cdot \left\{ \sum_{j=2}^{N/2} (\varepsilon_{[j]} - \varepsilon_{[1]}) \right\}^{-1}$$

After developing the terms of Eq. (13) this turns out to be:

$$f^-(\mathbf{r}) \simeq \sum_{i=2}^{N/2} \chi_{[i]}^- \rho_{[i]}(\mathbf{r})$$

$$f^-(\mathbf{r}) \simeq \frac{\varepsilon_{[2]} - \varepsilon_{[1]}}{(\varepsilon_{[2]} - \varepsilon_{[1]}) + (\varepsilon_{[3]} - \varepsilon_{[1]}) + \dots + (\varepsilon_{[N/2]} - \varepsilon_{[1]})} \cdot \rho_{[2]}(\mathbf{r}) + \dots$$

$$\dots + \frac{\varepsilon_{[N/2]} - \varepsilon_{[1]}}{(\varepsilon_{[2]} - \varepsilon_{[1]}) + (\varepsilon_{[3]} - \varepsilon_{[1]}) + \dots + (\varepsilon_{[N/2]} - \varepsilon_{[1]})} \cdot \rho_{[N/2]}(\mathbf{r})$$

And each  $\chi_{[i]}^-$  coefficient satisfies the condition  $0 < \chi_{[i]}^- < 1$  thus leading to the restriction for the sum of all coefficients of Eq. (13) with  $i$  ranging from 2 to  $N/2$ :

$$\sum_{i=2}^{N/2} \chi_{[i]}^- = 1$$

This also allows the normalization condition given by Eq. (2) to be satisfied. Furthermore, the following relationship will always be satisfied:

$$0 < \frac{\varepsilon_{[2]} - \varepsilon_{[1]}}{\sum_{j=2}^{N/2} (\varepsilon_{[j]} - \varepsilon_{[1]})} < \frac{\varepsilon_{[3]} - \varepsilon_{[1]}}{\sum_{j=2}^{N/2} (\varepsilon_{[j]} - \varepsilon_{[1]})} < \dots$$

$$\dots < \frac{\varepsilon_{[N/2-1]} - \varepsilon_{[1]}}{\sum_{j=2}^{N/2} (\varepsilon_{[j]} - \varepsilon_{[1]})} < \frac{\varepsilon_{[N/2]} - \varepsilon_{[1]}}{\sum_{j=2}^{N/2} (\varepsilon_{[j]} - \varepsilon_{[1]})} < 1,$$

As in the previous subsection, Eq.(13) is adapted intrinsically to the molecule's symmetry due to  $\chi_{[i]}^-$  coefficients. Together with that proposed in 2009 [24], Eq. (6) also corresponds to approximations that come from Eq. (13), demonstrated as follows:

Demonstration: imagine a molecular system with a degree of degeneration,  $q$ , in its occupied molecular orbitals; specifically, it presents a  $q$ -fold degenerated HOMO. That means:

$$\varepsilon_{[N/2]} - \varepsilon_{[1]} = \varepsilon_{[N/2-1]} - \varepsilon_{[1]} = \dots = \varepsilon_{[N/2+1-q]} - \varepsilon_{[1]} = \varepsilon^- :$$

$$f^-(\mathbf{r}) \simeq \sum_{i=2}^{N/2} \chi_{[i]}^- \rho_{[i]}(\mathbf{r})$$

$$\simeq \sum_{i=2}^{N/2-q} \chi_{[i]}^- \rho_{[i]}(\mathbf{r}) + \sum_{i=N/2+1-q}^{N/2} \chi_{[i]}^- \rho_{[i]}(\mathbf{r})$$

$$\simeq \sum_{i=2}^{N/2-q} \chi_{[i]}^- \rho_{[i]}(\mathbf{r}) + \frac{\varepsilon_{[N/2+1-q]} - \varepsilon_{[1]}}{\sum_{j=2}^{N/2} (\varepsilon_{[j]} - \varepsilon_{[1]})} \cdot \rho_{[N/2+1-q]}(\mathbf{r}) + \dots + \frac{\varepsilon_{[N/2]} - \varepsilon_{[1]}}{\sum_{j=2}^{N/2} (\varepsilon_{[j]} - \varepsilon_{[1]})} \cdot \rho_{[N/2]}(\mathbf{r})$$

$$\simeq \sum_{i=2}^{N/2-q} \chi_{[i]}^- \rho_{[i]}(\mathbf{r}) + \frac{\varepsilon^-}{\sum_{j=2}^{N/2} (\varepsilon_{[j]} - \varepsilon_{[1]})} \cdot \rho_{[N/2+1-q]}(\mathbf{r}) + \dots + \frac{\varepsilon^-}{\sum_{j=2}^{N/2} (\varepsilon_{[j]} - \varepsilon_{[1]})} \cdot \rho_{[N/2]}(\mathbf{r})$$

$$\simeq \sum_{i=2}^{N/2-q} \chi_{[i]}^- \rho_{[i]}(\mathbf{r}) + \frac{\varepsilon^-}{\sum_{j=2}^{N/2} (\varepsilon_{[j]} - \varepsilon_{[1]})} \cdot \sum_{i=N/2+1-q}^{N/2} \rho_{[i]}(\mathbf{r})$$

The denominator of the second term can also be split into two parts:

$$\sum_{j=2}^{N/2} (\varepsilon_{[j]} - \varepsilon_{[1]}) = \sum_{j=2}^{N/2-q} (\varepsilon_{[j]} - \varepsilon_{[1]}) + \sum_{j=N/2+1-q}^{N/2} (\varepsilon_{[j]} - \varepsilon_{[1]})$$

$$= \sum_{j=2}^{N/2-q} (\varepsilon_{[j]} - \varepsilon_{[1]}) + q \cdot \varepsilon^-$$

In consequence:

$$f^-(\mathbf{r}) \simeq \sum_{i=2}^{N/2-q} \chi_{[i]}^- \rho_{[i]}(\mathbf{r}) + \varepsilon^- \cdot \left\{ \sum_{j=2}^{N/2} (\varepsilon_{[j]} - \varepsilon_{[1]}) \right\}^{-1} \cdot \sum_{i=N/2+1-q}^{N/2} \rho_{[i]}(\mathbf{r})$$

$$\simeq \sum_{i=2}^{N/2-q} \chi_{[i]}^- \rho_{[i]}(\mathbf{r}) + \varepsilon^- \cdot \left\{ \sum_{j=2}^{N/2-q} (\varepsilon_{[j]} - \varepsilon_{[1]}) + q \cdot \varepsilon^- \right\}^{-1} \cdot \sum_{i=N/2+1-q}^{N/2} \rho_{[i]}(\mathbf{r})$$

Similarly, after considering the approximations  $\chi_{[i]}^- \simeq 0$  when  $i$  ranges from 2 to  $N/2-q$  along with  $\sum_{j=2}^{N/2-q} (\varepsilon_{[j]} - \varepsilon_{[1]}) \ll q \cdot \varepsilon^-$ , the electrophilic Fukui function adapted to the symmetry according to the degree of degeneration reported in 2009 is re-obtained [24]:

$$f^-(\mathbf{r}) \simeq \frac{1}{q} \sum_{i=N/2+1-q}^{N/2} \rho_{[i]}(\mathbf{r}) \equiv \frac{1}{q} \sum_{k=1}^q \rho_{\text{HOMO}_k}(\mathbf{r}).$$

As can be observed, this equation corresponds to that reported in 2009 [24], which is an approximation of Eq. (13) in case of a  $q$ -fold degenerated HOMO and when  $q=1$ , the classical expression given by Eq. (6) is recovered because  $\rho_{\text{HOMO}_1}(\mathbf{r}) \equiv \rho_{\text{HOMO}}(\mathbf{r})$ .

Dual descriptor as an expansion of electronic densities of occupied and virtual molecular orbitals

Equations (11) and (13) allow us to propose a new, more accurate, working equation for the DD:

$$f^{(2)}(\mathbf{r}) \simeq \sum_{i=N/2+1}^{K-1} \chi_{[i]}^+ \rho_{[i]}(\mathbf{r}) - \sum_{i=2}^{N/2} \chi_{[i]}^- \rho_{[i]}(\mathbf{r}) \quad (14)$$

This expression satisfies the restriction given by Eq. (8).

Based on the mathematical demonstrations concerning the degeneration phenomenon that can appear in FMOs as discussed above, it is very clear that the mathematical expression given by [24]:

$$f^{(2)}(\mathbf{r}) \simeq \frac{1}{p} \sum_{k=1}^p \rho_{\text{LUMO}_k}(\mathbf{r}) - \frac{1}{q} \sum_{k=1}^q \rho_{\text{HOMO}_k}(\mathbf{r}),$$

is an approximation of Eq. (14). The classical expression given by Eq. (10) is recovered when  $p=q=1$  because  $\rho_{\text{LUMO}_1}(\mathbf{r}) \equiv \rho_{\text{LUMO}}(\mathbf{r})$  and  $\rho_{\text{HOMO}_1}(\mathbf{r}) \equiv \rho_{\text{HOMO}}(\mathbf{r})$ .

### Dual descriptor expanded in terms of electronic densities of molecular orbitals. Open-shell systems

Like the closed-shell case, in open-shell systems there will be a basis set of  $K_\alpha$  molecular orbitals of  $\alpha$  type and  $K_\beta$  molecular orbitals of  $\beta$  type as depicted in Fig. 2. The system is going to contain  $N=N_\alpha+N_\beta$  electrons, thus meaning  $N_\alpha$  and  $N_\beta$  occupied molecular orbitals of  $\alpha$  and  $\beta$  type, respectively. The criteria used to choose proper expressions for the nucleophilic and electrophilic Fukui functions to use in open-shell systems are described in more detail in [13].

Within this context, as explained in [13], the  $i$  counter will range from 1 ( $\alpha$ -LOMO) to  $N_\alpha$  ( $\alpha$ -HOMO); after  $\alpha$ -HOMO, the  $i$  counter will take into account virtual  $\beta$  molecular orbitals ranging from  $N_\beta+1$  ( $\beta$ -LUMO) to  $K_\beta$  ( $\beta$ -HUMO).

Then  $\sum_{i=1}^{N_\alpha} \rho_{[i]}^\alpha(\mathbf{r}) + \sum_{i=1}^{N_\beta} \rho_{[i]}^\beta(\mathbf{r}) = \rho_N(\mathbf{r})$ , the total electronic density of the system conformed by  $N = N_\alpha + N_\beta$  electrons. In addition,

$$\int \rho_{[i]}^\alpha(\mathbf{r}) d\mathbf{r} = 1; \forall i \in \{1 \cdots K_\alpha\}$$

$$\int \rho_{[i]}^\beta(\mathbf{r}) d\mathbf{r} = 1; \forall i \in \{1 \cdots K_\beta\}$$

Nucleophilic Fukui function as an expansion of electronic densities of virtual molecular orbitals

Following an identical procedure to that established to define the mathematical expression of the nucleophilic Fukui function in closed-shell systems, in the case of open-shell systems, the following equation is suggested:

$$f_{\Delta N_S < 0}^+(\mathbf{r}) \simeq \sum_{i=N_\beta+1}^{K_\beta-1} \chi_{[i],\beta}^+ \rho_{[i]}^\beta(\mathbf{r}) \quad (15)$$

where

$$\chi_{[i],\beta}^+ = \left( \varepsilon_{[K_\beta]}^\beta - \varepsilon_{[i]}^\beta \right) \cdot \left\{ \sum_{j=N_\beta+1}^{K_\beta-1} \left( \varepsilon_{[K_\beta]}^\beta - \varepsilon_{[j]}^\beta \right) \right\}^{-1}.$$

Please refer to [13] for a full explanation of the notation  $\Delta N_S < 0$ . Note that the label  $[N_\beta+1]$  corresponds to the  $\beta$ -LUMO and  $[K_\beta]$  is the very last virtual (or unoccupied) molecular orbital ( $\beta$ -HUMO). The energy of  $\beta$ -HUMO,  $\varepsilon_{[K_\beta]}^\beta$ , is used as a reference value.

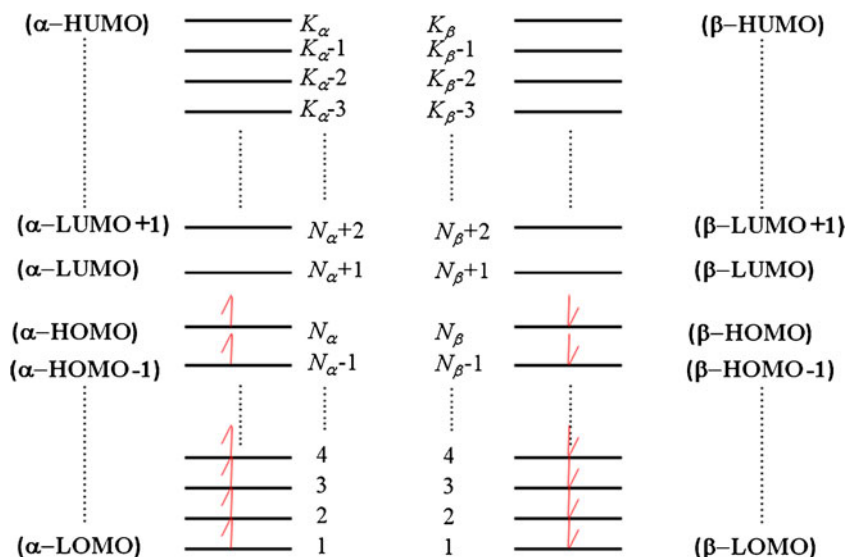
The following relationship will always be satisfied:

$$1 > \frac{\varepsilon_{[K_\beta]}^\beta - \varepsilon_{[N_\beta+1]}^\beta}{\sum_{j=N_\beta+1}^{K_\beta-1} \left( \varepsilon_{[K_\beta]}^\beta - \varepsilon_{[j]}^\beta \right)} > \cdots > \frac{\varepsilon_{[K_\beta]}^\beta - \varepsilon_{[K_\beta-1]}^\beta}{\sum_{j=N_\beta+1}^{K_\beta-1} \left( \varepsilon_{[K_\beta]}^\beta - \varepsilon_{[j]}^\beta \right)} > 0$$

The restriction  $0 < \chi_{[i],\beta}^+ < 1$  turns out to be in the following restriction for all  $i$  ranging from  $N_\beta+1$  to  $K_\beta-1$ :

$$\sum_{i=N_\beta+1}^{K_\beta-1} \chi_{[i],\beta}^+ = 1,$$

**Fig. 2** Diagram of molecular orbital energies in an open-shell system



so that the normalization condition given by Eq. (2) is satisfied. Since the definition of Eq. (15) is the same mathematically as the definition given by Eq. (11), no analysis of coefficients  $\chi_{[i,\beta]}^+$  and degeneracies on  $\beta$  frontier molecular orbitals is needed in this subsection.

Electrophilic Fukui function as an expansion of electronic densities of virtual molecular orbitals

Similarly to electrophilic Fukui function in closed-shell systems, the proposed equation for open-shell systems is as follows:

$$f_{\Delta N_S < 0}^-(\mathbf{r}) \simeq \sum_{i=2}^{N_\alpha} \chi_{[i,\alpha]}^- \rho_{[i]}^\alpha(\mathbf{r}) \tag{16}$$

where

$$\chi_{[i,\alpha]}^- = \left( \varepsilon_{[i]}^\alpha - \varepsilon_{[1]}^\alpha \right) \cdot \left\{ \sum_{j=2}^{N_\alpha} \left( \varepsilon_{[j]}^\alpha - \varepsilon_{[1]}^\alpha \right) \right\}^{-1}$$

The energy of  $\alpha$ -LOMO,  $\varepsilon_{[1]}^\alpha$ , is used as a reference value. The following relationship is satisfied:

$$0 < \frac{\varepsilon_{[2]}^\alpha - \varepsilon_{[1]}^\alpha}{\sum_{j=2}^{N_\alpha} \left( \varepsilon_{[j]}^\alpha - \varepsilon_{[1]}^\alpha \right)} < \frac{\varepsilon_{[3]}^\alpha - \varepsilon_{[1]}^\alpha}{\sum_{j=2}^{N_\alpha} \left( \varepsilon_{[j]}^\alpha - \varepsilon_{[1]}^\alpha \right)} < \dots$$

$$\dots < \frac{\varepsilon_{[N_\alpha-1]}^\alpha - \varepsilon_{[1]}^\alpha}{\sum_{j=2}^{N_\alpha} \left( \varepsilon_{[j]}^\alpha - \varepsilon_{[1]}^\alpha \right)} < \frac{\varepsilon_{[N_\alpha]}^\alpha - \varepsilon_{[1]}^\alpha}{\sum_{j=2}^{N_\alpha} \left( \varepsilon_{[j]}^\alpha - \varepsilon_{[1]}^\alpha \right)} < 1$$

And each coefficient satisfies the condition:  $0 < \chi_{[i]}^- < 1$ . Along with the normalization condition of the Fukui function given by Eq. (2), this provides the restriction for the sum of all coefficients of Eq. (16):

$$\sum_{i=2}^{N_\alpha} \chi_{[i,\alpha]}^- = 1$$

Dual descriptor as an expansion of electronic densities of occupied and virtual molecular orbitals

The corresponding equation is:

$$f_{\Delta N_S < 0}^{(2)}(\mathbf{r}) \simeq \sum_{i=N_\beta+1}^{K_\beta-1} \chi_{[i,\beta]}^+ \rho_{[i]}^\beta(\mathbf{r}) - \sum_{i=2}^{N_\alpha} \chi_{[i,\alpha]}^- \rho_{[i]}^\alpha(\mathbf{r}) \tag{17}$$

, which satisfies the restriction given by Eq. (8).

Although there were no mathematical demonstrations carried out in the subsections above in order to show that the equations of the nucleophilic and electrophilic Fukui functions adapted to the molecular symmetry of open-shell systems as reported in a previous article in 2011 [13] correspond to approximations of Eqs. (15) and (16), respectively, the algebraic analysis would be the same as was exposed here, so it is also possible to state that the mathematical expression that appears in [13]:

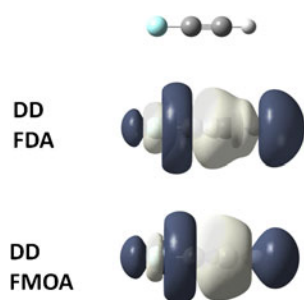
$$f_{\Delta N_S < 0}^{(2)}(\mathbf{r}) \simeq \frac{1}{p_\beta} \sum_{k=1}^{p_\beta} \rho_{\text{LUMO}_k}^\beta(\mathbf{r}) - \frac{1}{q_\alpha} \sum_{k=1}^{q_\alpha} \rho_{\text{HOMO}_k}^\alpha(\mathbf{r}),$$

is an approximation of Eq. (17) as well.

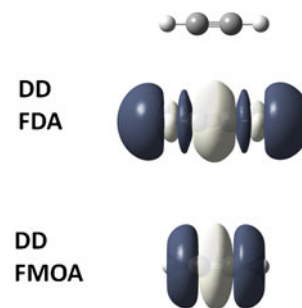
## Applications

The emphasis will be focused on closed-shell systems. Acetylene (or ethyne;  $\text{H}-\text{C}\equiv\text{C}-\text{H}$ ) and fluoroethyne ( $\text{F}-\text{C}\equiv\text{C}-\text{H}$ ) molecules were optimized geometrically without symmetry restrictions according to the Schlegel algorithm [36] at the DFT level of theory. The functionals used in these calculations were the Becke-3 for exchange and Lee-Yang-Parr for correlation [37–40]. The basis set 6–31 G(d) [41–43] was used. The frequency calculation was performed at the same level of theory in order to ensure that each optimized structure corresponds to an energy minimum. All calculations were carried out using GAUSSIAN 09 [44] code. 3-D maps of molecular electrostatic potential (MEP) and DD were generated by using the cubegen complementary code of the GAUSSIAN 09 software package.

When the FMOA is accurate enough, the DD is well represented as depicted by Fig. 3, where  $\text{F}-\text{C}\equiv\text{C}-\text{H}$  is used as an example. This molecule belongs to the  $C_{\infty h}$  point group symmetry and, as can be observed, any 3-D map of the DD must belong to the  $A_1$  totally symmetric irreducible representation. The finite difference approximation (FDA) allows one to realize that FMOA gives an acceptable 3-D map of the DD. On the other hand, the  $\text{H}-\text{C}\equiv\text{C}-\text{H}$  molecule belongs to the  $D_{\infty h}$  point group symmetry; consequently, any 3-D map of the DD must also belong to the  $\Sigma_g^+$  totally symmetric irreducible representation. Thus all 3-D maps satisfy this symmetry requirement, but this molecule reveals that FMOA differs noticeably from the FDA as depicted in Fig. 4. This might be a consequence of the orbital relaxation not taken into account by the FMOA. The aim of this section is to demonstrate that the intermediate level of

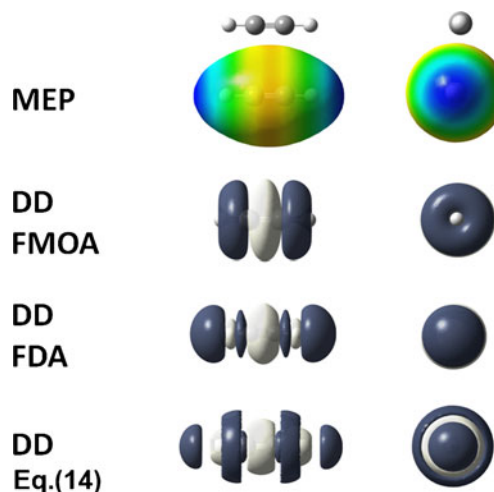


**Fig. 3** Dual descriptor (DD) of fluoroethyne ( $\text{F}-\text{C}\equiv\text{C}-\text{H}$ ). *FDA* Finite difference approximation, *FMOA* frontier molecular orbital approximation. The light-blue colored atom is fluorine; the carbon atoms are depicted in grey color and the white colored atom is hydrogen. Dual descriptor (DD),  $f^{(2)}(\mathbf{r})$  is represented as a 3-D map of two colors: when  $f^{(2)}(\mathbf{r}) > 0$  implies a dark colored lobe thus indicating that nucleophilic attacks will be oriented toward there; when  $f^{(2)}(\mathbf{r}) < 0$  implies a light colored lobe thus indicating that electrophilic attacks will be oriented toward there. Two different ways to obtain DD have been depicted in this figure: the first row shows the FDA; the second one shows the FMOA. All isosurfaces have been depicted at 0.001 a.u.



**Fig. 4** DD of ethyne or acetylene ( $\text{H}-\text{C}\equiv\text{C}-\text{H}$ ). Dual descriptor (DD),  $f^{(2)}(\mathbf{r})$  is represented as a 3-D map of two colors: when  $f^{(2)}(\mathbf{r}) > 0$  implies a dark colored lobe thus indicating that nucleophilic attacks will be oriented toward there; when  $f^{(2)}(\mathbf{r}) < 0$  implies a light colored lobe thus indicating that electrophilic attacks will be oriented toward there. Two different ways to obtain DD have been depicted in this figure: the first row shows the FDA; the second one shows the FMOA. All isosurfaces have been depicted at 0.001 a.u.

approximation proposed here to calculate the DD should provide a similar 3-D map to that depicted by FDA provided that the FDA is an exact representation of the DD. Consequently, the acetylene molecule will be used as a test molecule in order to visualize the result of the DD built from the approximation proposed here for closed-shell systems through the use of Eq. (14).



**Fig. 5** Front (*left*) and side (*right*) views of the acetylene molecule along with its respective molecular electrostatic potential (MEP) (*top row*) and DD (*lower rows*). The MEP is represented as a 3-D map ranging from negatives values (*red*) to positive values (*blue*), that mean  $\{V(\mathbf{r})\}_{\min} \leq V(\mathbf{r}) \leq \{V(\mathbf{r})\}_{\max}$  where  $\{V(\mathbf{r})\}_{\min} = -4.405 \cdot 10^{-2}$  a.u. and  $\{V(\mathbf{r})\}_{\max} = 4.405 \cdot 10^{-2}$  a.u. A *red colored zone* indicates a dominant effect of electrons (electrically negative), *green colored zones* indicate sites where the effects of nuclei and electrons are balanced, and a *blue colored zone* shows sites where the effect of nuclei is dominant (electrically positive). DD,  $f^{(2)}(\mathbf{r})$  is represented as a 3-D map of two colors: *dark colored lobe*  $f^{(2)}(\mathbf{r}) > 0$  implying that nucleophilic attacks will be oriented here; *light colored lobe*  $f^{(2)}(\mathbf{r}) < 0$  implying that electrophilic attacks will be oriented here. Three different ways of obtaining DD are depicted: *second row* FMOA; *third row* FDA, *bottom row* proposed way to depict DD. All isosurfaces have been depicted at 0.001 a.u.



A closed-shell system like acetylene requires the use of Eq. (14) to generate a 3-D map of the DD. All coefficients  $\chi_{[i]}^+$  and  $\chi_{[i]}^-$  were then generated by hand and electronic densities of occupied and virtual molecular orbitals were computed through the use of the cubegen and cubman computational codes coming from the GAUSSIAN 09 [44] code. After using this equation, Fig. 5 allows the FMOA, the FDA and the present intermediate level of approximation to be compared. As can be observed, all approximations satisfy the symmetry requirements. The MEP, which is an observable, has been used as a function to verify symmetry requirements. This molecule presents a two-fold HOMO and a two-fold LUMO, respectively.

Figure 5 shows a similarity between the DD generated by means of Eq. (14) and that generated by FDA, thus validating the approach proposed in this article. The DD obtained under the FMOA provides only one negative phase located around the triple bond, meanwhile Eq. (14) generates a DD with more than one negative region. This means that, in addition to the surroundings of the triple bond, there are small negative regions that are localized near the hydrogen atoms, but in general electrophilic attacks are preferred on the triple bond. The MEP also indicates that electrophilic attacks are preferred to be carried out on the triple bond, so that electrophilic attacks on the acetylene molecule are supported by electrostatic and covalent reasons. However, a difference between the shape of the DD based on the FMOA and that based on Eq. (14) as proposed here can be noted.

Continuing with the analysis, Eq. (14) reveals positive regions localized over hydrogen atoms along axial axis, thus implying that nucleophilic attacks could be oriented along the axial axis of acetylene towards hydrogen atoms, this is not seen when using the FMOA. Note that the MEP also reveals that nucleophilic attacks will be oriented toward hydrogen atoms along the same axial axis thus indicating that nucleophilic attacks on hydrogen atoms are also supported by covalent and electrostatic reasons.

## Conclusions

Working Eqs. (14) and (17) have been proposed here to obtain a much better representation of the DD than with conventional equations based solely on FMO densities because the influence of the electronic densities of the remaining molecular orbitals has been included by means of two types of coefficients:  $\chi_{[i]}^+$  and  $\chi_{[i]}^-$  in the case of closed-shell systems and  $\chi_{[\beta,\delta]}^+$  and  $\chi_{[\beta,\alpha]}^-$  in the case of open-shell systems. In addition, these equations have the advantage that any possible degeneracy or quasi-degeneracy in FMOs is taken into account without the user needing to perform a previous analysis. Equations (14) and (17) are useful for all types of

molecular systems, whether or not they exhibit FMO degeneracies, and the reader is invited to prove these equations on molecular systems such as diatomic molecules, boranes, fullerenes, and even systems based on transition metals that exhibit quasi-degeneration in their molecular orbitals. Such degeneration is no longer a concern for the user because, unlike equations reported in the literature [13, 24] where the user would have to take into account explicitly the degree of degeneration with the aim of assigning the correct number for  $p$  and  $q$  in a closed-shell system (or  $p_\beta$  and  $q_\alpha$  in an open-shell system), Eqs. (14) and (17) are able to take into account automatically the phenomenon of degeneration with the advantage of including the contribution to local reactivity of the remaining molecular orbitals (except LOMO and HUMO) to properly describe the local reactivity of the system under study.

Since open-shell systems imply a set of working equations that combine four possible configurations ( $\alpha-\alpha$ ,  $\alpha-\beta$ ,  $\beta-\alpha$  and  $\beta-\beta$ ), Eq. (17), corresponding to the  $\beta-\alpha$  and three remaining configuration (not examined in this article) should be analyzed in future works.

Finally, a general warning that the use of the approaches proposed in this article for closed- and open-shell systems is based on the fact that the nucleophilic Fukui function is strongly basis-set-dependent because the HUMO energy diverges strongly as the size of the basis set decreases; therefore, some kind of mathematical strategy is required to deplete such basis set dependence. This task is ongoing.

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